

AN INVOLUTORIAL SPACE TRANSFORMATION ASSOCIATED
WITH A $Q_{1,n}$ CONGRUENCE

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BY M. L. VEST

1. **Introduction.** Recently the present author [2] gave an analytical discussion of a non-involutorial transformation associated with the congruence of lines on a plane curve of order n having an $(n - 1)$ -point and a secant through that point. In what follows an involutorial transformation associated with such a congruence is studied, the procedure again being almost entirely analytical.

DePaolis [1] some years ago published a synthetic discussion of four particular involutorial transformations associated with the congruence of lines which are bisecants of a space curve of order μ and a chord meeting the curve in $\mu - 1$ points. Of these transformations the "second species" may be considered to be, in a sense, the most general since for a space curve of given order and for given multiplicities of this curve and of the bisecant upon the pointwise invariant surface the order of this transformation is higher than those of the other three types. The present transformation is equivalent to this "second species" from the standpoint of resemblance in those properties which DePaolis described. No essential properties of the transformation have been lost through the use of a plane curve while the analytical treatment gives many details not mentioned in [1] (such as the existence of parasitic lines) and points out several errors, the order given for his curve Γ being a case in point.

The bundle of lines through the multiple point is not considered as belonging to the congruence and the tangents to the curve at this point are first assumed to be distinct.

Given the plane n -ic r , a line s meeting r at an $(n - 1)$ -point A , and a pencil of surfaces

$$|F_m| : s^{m-2} g_{4m-4}$$

which, as indicated, are of order m and contain s $(m - 2)$ -times. Through a generic point $P(y)$ there passes a single F of $|F|$. The unique line t belonging to the congruence and passing through $P(y)$ meets F a second time in one residual point $Q(x)$, image of $P(y)$ under the transformation thus defined. The residual base curve of $|F|$ has been denoted by g , is of order $4m - 4$, and is considered non-composite. (This curve corresponds to DePaolis' curve Γ whose order seems to be incorrectly given). It will be shown that r , s and g are fundamental curves of the involution and that A is a fundamental point of the second kind.

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2. **Equations of the transformation.** Let us take the equations of r and s , respectively, as

$$(1) \quad \begin{aligned} x_3(cx_1 x_2) - (dx_1 x_2) &= 0, & x_4 &= 0; \\ x_1 &= x_2 = 0, \end{aligned}$$

where

$$(2) \quad (cx_1 x_2) = \sum_{k=0}^{n-1} c_{k, n-k-1} x_1^k x_2^{n-k-1}, \quad (dx_1 x_2) = \sum_{h=0}^n d_{h, n-h} x_1^h x_2^{n-h},$$

and the pencil of surfaces $|F|$ as

$$(3) \quad |F_m| \equiv U - \lambda U' = 0,$$

where

$$(4) \quad \begin{aligned} U_m &= [ax]\{ex_1 x_2\} - [\alpha x]\{\epsilon x_1 x_2\}, \\ U'_m &= [a'x]\{e'x_1 x_2\} - [\alpha'x]\{\epsilon'x_1 x_2\}, \\ [ax] &= \sum_{i,j=1}^4 a_{ij} x_i x_j, & [a'x] &= \sum_{i,j=1}^4 a'_{ij} x_i x_j, \\ \{ex_1 x_2\} &= \sum_{p=0}^{m-2} e_{p, m-p-2} x_1^p x_2^{m-p-2}, \\ \{e'x_1 x_2\} &= \sum_{p=0}^{m-2} e'_{p, m-p-2} x_1^p x_2^{m-p-2}, \end{aligned}$$

and so on.

Through a generic point $P(y)$ there is an F of $|F|$ with parameter $\lambda = U(y)/U'(y)$ and equation

$$(5) \quad U(x)U'(y) - U(y)U'(x) = 0.$$

The unique transversal t through $P(y)$, r and s meets the surface (5) a second time in a point $Q(x)$ having coordinates

$$(6) \quad \begin{aligned} \tau x_1 &= Ry_1 + K(cy_1 y_2)y_1 = Sy_1, \\ \tau x_2 &= Ry_2 + K(cy_1 y_2)y_2 = Sy_2, \\ \tau x_3 &= Ry_3 + K(dy_1 y_2), \\ \tau x_4 &= Ry_4, \end{aligned}$$

where

$$R_{2m+2n-2} = UW' - U'W, \quad K_{2m+n-1} = U'V - UV',$$

$$S_{2m+2n-2} = R + K(cy_1 y_2),$$

$$W_{m+2n-2} = C\{ey_1 y_2\} - \Gamma\{\epsilon y_1 y_2\}, \quad W'_{m+2n-2} = C'\{e'y_1 y_2\} - \Gamma'\{\epsilon' y_1 y_2\},$$

$$V_{m+n-1} = D\{ey_1 y_2\} - \Delta\{\epsilon y_1 y_2\}, \quad V'_{m+n-1} = D'\{e'y_1 y_2\} - \Delta'\{\epsilon' y_1 y_2\},$$

$$C_{2n} = (cy_1 y_2)^2(a_{11} y_1^2 + 2a_{12} y_1 y_2 + a_{22} y_2^2)$$

$$(7) \quad + 2(cy_1 y_2)(dy_1 y_2)(a_{13} y_1 + a_{23} y_2) + a_{33}(dy_1 y_2)^2,$$

$$\Gamma_{2n} = (cy_1 y_2)^2(\alpha_{11} y_1^2 + 2\alpha_{12} y_1 y_2 + \alpha_{22} y_2^2)$$

$$+ 2(cy_1 y_2)(dy_1 y_2)(\alpha_{13} y_1 + \alpha_{23} y_2) + \alpha_{33}(dy_1 y_2)^2,$$

$$D_{n+1} = (cy_1 y_2)[(a_{11} y_1^2 + 2a_{12} y_1 y_2 + a_{22} y_2^2) + y_3(a_{13} y_1 + a_{23} y_2)$$

$$+ y_4(a_{14} y_1 + a_{24} y_2)] + (dy_1 y_2)(a_{13} y_1 + a_{23} y_2 + a_{33} y_3 + a_{43} y_4),$$

and so on.

Equations (6) are those of the involution under consideration.

3. Images of fundamental curves and elements. The transformation I applied to an F of $|F|$ gives

$$(8) \quad U \sim (I)US^{m-2}G,$$

where

$$G_{4m+4n-4} = R^2 + 2K(W'V - WV').$$

Here $r \sim (I)R$, $s \sim (I)S$, $g \sim (I)G$. Similarly

$$R \sim (I)RS^{2m+2n-4}G, \quad S \sim (I)R^2S^{2m+2n-5}G,$$

$$(9) \quad G \sim (I)R^4S^{4m+4n-8}G, \quad K \sim (I)KRS^{2m+n-4}G.$$

Through a point O_r on r there is a pencil of transversals through s . O_r determines an F which cuts the pencil in s and a conic k . As O_r describes r the conic k generates the surface R , image of r .

Through a point O_s on s there is an n -ic cone of transversals to r , to each line of which corresponds one F of $|F|$ meeting that line in one residual point. The locus of such points is a curve c which generates the surface S , image of s . The order of c , determined by the intersection of S and a homaloidal surface, is $2m + 2n - 4$.

Through a point O_g on g there is a unique line t of the congruence. However, every F of $|F|$ contains O_g , hence $O_g \sim (I)t$. The ruled surface G , generated by t , is the image of g under I .

Let us designate a point common to s and g as O_{sg} . The image of such a point, since it lies on s , is a $c_{2m+2n-4}$. However, since the point also lies on g , the $c_{2m+2n-4}$ must contain a line l , hence is composite. The number of such lines l , i.e., the number of intersections of s and g , is shown by the intersection of S and G to be $4m - 8$.

The multiple point A is a fundamental point of the second kind having as its image $n - 1$ conics $f_{2,1}, \dots, f_{2,n-1}$, one lying in each of the $n - 1$ planes determined by s and the tangent lines to r at A .

4. Contact along s , g and r . The equations of the line s may be written as $x_1 = 0, x_2 = 0, x_3 = k, x_4 = 1$. Let us set up a projectivity which will carry a generic point of s into a vertex of the coordinate tetrahedron, say in the manner

$$(1, 0, 0, 0) \rightarrow (1, 0, 0, 0), \quad (0, 1, 0, 0) \rightarrow (0, 1, 0, 0)$$

$$(0, 0, k, 1) \rightarrow (0, 0, 1, 0), \quad (0, 0, 0, 1) \rightarrow (0, 0, 0, 1).$$

The required projectivity takes the form $y_1 = x_1, y_2 = x_2, y_3 = kx_3, y_4 = x_3 + x_4$.

When this projectivity is applied to the fundamental surfaces of the transformation it is found that the coefficients of the highest powers of x_3 in the transformed equations of S and K have a common factor of order $2m + n - 4$ which is

$$(L'\{e'y_1 y_2\} - \Lambda'\{\epsilon'y_1 y_2\})(T\{ey_1 y_2\} - \Theta\{\epsilon y_1 y_2\}) \\ - (L\{ey_1 y_2\} - \Lambda\{\epsilon y_1 y_2\})(T'\{e'y_1 y_2\} - \Theta'\{\epsilon'y_1 y_2\}),$$

where

$$L = a_{33} k^2 + 2a_{34} k + a_{44}, \quad \Lambda = \alpha_{33} k^2 + 2\alpha_{34} k + \alpha_{44}$$

$$T = (cy_1 y_2)[y_1(a_{13} k + a_{14}) + y_2(a_{23} k + a_{24})] + (dy_1 y_2)(a_{33} k + a_{34})$$

and so on. It follows that S and K have contact of order $2m + n - 4$ along s .

To investigate the existence of tangency between the fundamental surfaces along the base curve g , the fundamental system is cut by an arbitrary plane Π containing s . Since s intersects r in $n - 1$ points and g in $4m - 8$ points, Π meets r in one residual point O_r and g in four residual points $\Gamma_1, \dots, \Gamma_4$. The pencil $|F|$ does not contain r ; hence O_r is, in general, distinct from the points Γ_i which are in turn distinct points since g is non-composite. Through O_r and each point Γ_i there is a line γ_i of the congruence which, since Γ_i lies on every F of $|F|$, is the image of Γ_i , i.e., $\Gamma_i \sim (I)\gamma_i$, and γ_i is a generator of the surface

G. We note that, in general, there is only one point Γ_i on γ_i for otherwise γ_i would be a parasitic line of which there are only a finite number (see paragraph 5). It follows that $\Gamma_i \sim (I)\Gamma_i$, hence it is a point on κ , the section of the invariant surface *K*. The line γ_i cuts κ in $2m + n - 1 - (2m + n - 4) = 3$ points not on *s*. One of these is at Γ_i as we have seen, and one is at O_r since *r* lies on *K*. From (9) $K \sim (I) KRS^{2m+n-4}G$, showing that *r* is single upon *K*, so that κ cannot, in general, have a multiple point at O_r . Nor is κ , in general, tangent to γ_i at O_r for this would imply tangency between *K* and *G* along *r* whereas the intersection of *K* and *G* shows such tangency does not exist. The third intersection of κ and γ_i must then be at Γ_i for all remaining points of γ_i are carried into Γ_i by the involution while every point on κ is invariant. Furthermore, Γ_i is not a double point on κ for *g* occurs only once on *K* (see (9)) hence γ_i is tangent to κ at Γ_i and *G* and *K* have simple tangency along *g*.

The image of the point O_r consists of the line *s* counted $m - 2$ times and a conic *k* in the plane Π . The conic *k* contains O_r as an ordinary point for the surface *R*, generated by *k*, contains *r* once (see (9)). Furthermore, the points $\Gamma_1, \dots, \Gamma_4$ also lie upon *k* for every point of *k* must go into O_r and, since each point Γ_i is carried into the line γ_i by the involution, it goes into O_r . Because *s* lies $2m + n - 4$ times upon *K*, κ consists of the line *s* and a cubic *z* meeting the conic *k* in six points, five of which are $\Gamma_1, \dots, \Gamma_4$ and O_r as shown in the preceding paragraph. The cubic *z* meets *k* only once at each of these points since κ is tangent to the line γ_i at the corresponding point Γ_i . The sixth point of intersection is evidently at O_r since every point on *k* goes into O_r while every point on κ is invariant. Since O_r is not a multiple point on κ it follows that κ must be tangent to *k* at O_r , and when Π generates the pencil on *s*, the surfaces *R* and *K*, generated by *k* and κ , respectively, will have simple tangency along *r*.

5. Parasitic lines. Any line *p* of the congruence which meets *g* twice will be such that every point of the line is transformed into the entire line, i.e., is parasitic. Such a line is double on *G* and lies once upon *R*, *S*, *K* and each homaloidal surface ϕ . The number of parasitic lines, as determined by the intersection of two homaloidal surfaces, is $6m + 6n - 10$.

6. Invariant and homaloidal surfaces. It is evident from equations (6) that the surface $K = 0$ is pointwise invariant under *I*. The plane $x_4 = 0$ and all planes containing the line *s* are invariant but not pointwise invariant.

Applying the involution to a generic plane π gives

$$\pi \equiv (Ax) \sim (I)R(Ay) + K(Az) = \phi_{2m+2n-1},$$

where

$$(Ax) = \sum_{i=1}^4 A_i x_i, \quad (Az) = (A_1 y_1 + A_2 y_2)(cy_1 y_2) + A_3(dy_1 y_2).$$

The ϕ 's are homaloidal surfaces of the transformation. Further,

$$\phi \sim (I)(Ax)R^2S^{2m+2n-4}G,$$

hence the homaloidal web is

$$\infty^3 | \phi | : r^2 s^{2m+2n-4} g(6m+6n-10)p.$$

The intersection of two homaloidal surfaces gives the homaloidal net

$$H \equiv [\phi\phi] : r^4 s^{4m^2+4n^2+8mn-16m-16n+16} (6m+6n-10) p c_{2m+2n-1},$$

where $c_{2m+2n-1}$ is the image of the line $[\pi\pi]$.

The jacobian of the involution is $J_{sm-sn-s} = RSG$.

7. Table of characteristics. The images of planes and of fundamental elements may now be expressed by the following table:

$$\pi \sim (I)\phi : r^2 s^{2m+2n-4} g(6m+6n-10)p,$$

$$r \sim (I)R : r^{1+t} s^{2m+2n-4} g(6m+6n-10) p f_{2,1} \cdots f_{2,n-1},$$

$$s \sim (I)S : r^2 s^{2m+2n-3+(2m+n-4)t} g(4m-8)l(6m+6n-10) p f_{2,1} \cdots f_{2,n-1},$$

$$g \sim (I)G : r^4 s^{4m+4n-8} g^{1+t} (4m-8)l(6m+6n-10)p,$$

$$K \sim (I)K : r^{1+t} s^{2m+n-4+(2m+n-4)t} g^{1+t} (6m+6n-10)p,$$

where the coefficients of t in the multiplicities of r , s and g indicate the order of contact as found in paragraph 4.

8. Intersection table. The complete intersection table may now be written as follows:

$$[\phi R] : r^2 s^{4m^2+8mn+4n^2-16m-16n+16} g(6m+6n-10) p k_{2,1} \cdots k_{2,n},$$

$$[\phi S] : r^4 s^{4m^2+8mn+4n^2-18m-18n+20} g(6m+6n-10) p c_{2m+2n-4},$$

$$[\phi G] : r^8 s^{8m^2+16mn+8n^2-32m-32n+32} g^2(6m+6n-10) p(4m-4)t,$$

$$[\phi K] : r^2 s^{4m^2+6mn+2n^2-16m-12n+16} g(6m+6n-10) p f_{2,n-1},$$

$$[RS] : r^2 s^{4m^2+8mn+4n^2-18m-18n+20} g(6m+6n-10) p f_{2,1} \cdots f_{2,n-1},$$

$$[RG] : r^4 s^{8m^2+16mn+8n^2-32m-32n+32} g^2(6m+6n-10)p,$$

$$[RK] : r^{1+t} s^{4m^2+6mn+2n^2-16m-12n+16} g(6m+6n-10)p,$$

$$[SG] : r^8 s^{8m^2 + 16mn + 8n^2 - 36m - 36n + 40} g(4m - 8)t(6m + 6n - 10)p,$$

$$[SK] : r^2 s^{4m^2 + 6mn + 2n^2 - 18m - 13n + 20 + (2m + n - 4)t} g(6m + 6n - 10)p,$$

$$[GK] : r^4 s^{8m^2 + 12mn + 4n^2 - 32m - 24n + 32} g^{1+1t} 2(6m + 6n - 10)p.$$

The lines t , the conics $k_{2,1}, \dots, k_{2,n}$ and the curve $c_{2m+2n-4}$ are the images of the points of intersection of π with g , r and s , respectively, and the curve f_{2m+n-1} is the intersection $[\pi K]$.

9. Degeneracies. Throughout this paper the base curve g is considered to be non-composite. It may be pointed out that if g should become composite in such a way as to give rise to a ruled surface P of parasitic lines, quadrisecants of the system r , s and g , the order of the involution would be reduced by the order of P . The author hopes to study possible degeneracies of this transformation in a later paper.

10. The I_3 in a plane through s . A plane $\pi \equiv x_1 = \theta x_2$ cuts the surfaces of $|F|$ in a residual pencil of conics

$$(10) \quad |f_2| = u - \mu u' = 0,$$

where

$$(11) \quad \begin{aligned} u_2 &= [ax]\{e\theta\} - [\alpha x]\{\epsilon\theta\}, & u'_2 &= [a'x]\{e'\theta\} - [\alpha'x]\{\epsilon'\theta\}, \\ [ax] &= (a_{11}\theta^2 + 2a_{12}\theta + a_{22})x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 \\ &\quad + 2(a_{13}\theta + a_{23})x_2x_3 + 2(a_{14}\theta + a_{24})x_2x_4 + 2a_{34}x_3x_4, \\ \{e\theta\} &= \sum_{p=0}^{m-2} e_{p, m-p-2} \theta^p, \end{aligned}$$

and so on. The n -ic r intersects π in one residual point $P: [\theta(c\theta), (c\theta), (d\theta), 0]$ where

$$(c\theta) = \sum_{k=0}^{n-1} c_{k, n-k-1} \theta^k, \quad (d\theta) = \sum_{h=0}^n d_{h, n-h} \theta^h.$$

Through a generic point $P(y)$ of π passes an f of $|f|$ having parameter $\mu = u(y)/u'(y)$ and equation

$$(12) \quad u(x)u'(y) - u'(x)u(y) = 0.$$

The unique transversal t through $P(y)$ and P meets (12) again in one residual point $Q(x)$, image of $P(y)$ under the involution thus defined. The involution has as its equation

$$(13) \quad I_3: \quad x_2 = \rho y_2 + \kappa(c\theta), \quad x_3 = \rho y_3 + \kappa(d\theta), \quad x_4 = \rho y_4,$$

where

$$\begin{aligned}
 \rho_2 &= w'u - wu', & \kappa_3 &= 2(vu' - v'u), \\
 w &= b\{e\theta\} - \beta\{\epsilon\theta\}, & w' &= b'\{e'\theta\} - \beta'\{\epsilon'\theta\}, \\
 v &= d\{e\theta\} - \delta\{\epsilon\theta\}, & v' &= d'\{e'\theta\} - \delta'\{\epsilon'\theta\}, \\
 (14) \quad b &= (a_{11}\theta^2 + 2a_{12}\theta + a_{22})(c\theta)^2 + 2(a_{13}\theta + a_{23})(c\theta)(d\theta) + a_{33}(d\theta)^2, \\
 d &= [(a_{11}\theta^2 + 2a_{12}\theta + a_{22})y_2 + (a_{13}\theta + a_{23})y_3 + (a_{14}\theta + a_{24})y_4](c\theta) \\
 &\quad + [(a_{13}\theta + a_{23})y_2 + a_{33}y_3 + a_{34}y_4](d\theta),
 \end{aligned}$$

and so on.

The curve κ is pointwise invariant under the involution and $P \sim (I)\rho$.

Upon applying the involution we get

$$\begin{aligned}
 (15) \quad u &\sim (I)u\gamma, & \rho &\sim (I)\rho\gamma, & \kappa &\sim (I) - \kappa\rho\gamma, & \gamma &\sim (I)\rho^4\gamma, \\
 \gamma_4 &= \rho^2 - 2\kappa(v'w - vw').
 \end{aligned}$$

The base points of the pencil $|f|$ are the four points $\Gamma_1, \dots, \Gamma_4$. The lines $P\Gamma_1, \dots, P\Gamma_4$ are designated $\gamma_1, \dots, \gamma_4$, respectively. The curve γ in (15) consists of the lines $\gamma_1, \dots, \gamma_4$ while $\Gamma_i \sim (I)\gamma_i$.

When the involution is applied to a generic line we get

$$\begin{aligned}
 \lambda &\equiv a_2x_2 + a_3x_3 + a_4x_4 \sim (I)\rho\lambda + \kappa[a_2(c\theta) + a_3(d\theta)] = \phi_3, \\
 \phi &\sim (I)\lambda\rho^2\gamma,
 \end{aligned}$$

and the homaloidal net is

$$\infty^2 | \phi | : P^2\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4.$$

It has been shown in §4 that κ is tangent to each line γ_i at the point Γ_i and that κ is tangent to ρ at the point P .

The following table of characteristics and intersections has now been established:

$$\begin{aligned}
 \lambda &\sim (I)\phi : P^2\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4, & P &\sim (I)\rho : P^{1+1t}\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4, \\
 \Gamma_i &\sim (I)\gamma_i : P\Gamma_i^{1+1t}, & \kappa &\sim (I)\kappa : P^{1+1t}\Gamma_1^{1+1t}\Gamma_2^{1+1t}\Gamma_3^{1+1t}\Gamma_4^{1+1t}, \\
 [\phi\phi] &: P^4\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 B \text{ (the homaloidal web),} \\
 [\phi\rho] &: P^2\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4, & [\phi\gamma_i] &: P^2\Gamma_i, \\
 [\phi\kappa] &: P^2\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 Z_1 Z_2 Z_3, & [\rho\gamma_i] &: P\Gamma_i, \\
 [\rho\kappa] &: P^{1+1t}\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4, & [\gamma_i\kappa] &: P\Gamma_i^{1+1t}.
 \end{aligned}$$

The point B is the image of $[\lambda\lambda]$ and the points Z_1, Z_2, Z_3 are the intersections of λ and κ .

The jacobian is $\rho\gamma$.

As the plane π generates the pencil on s the I_3 generates the space $I_{2m+2n-1}$ whose equations may be obtained from (13) by replacing $u, u', v, v', w, w', \theta$ by $U, U', V, V', W, W', x_1/x_2$, respectively, and so on. Since the point P is the section of r by π , r may be represented by $x_1 = \theta(c\theta), x_2 = (c\theta), x_3 = (d\theta), x_4 = 0$.

11. r having $(n - 1)$ -point with coincident tangents. If k of the tangents to r at A are coincident the transformation will differ from the above with respect to the image of A , the correspondences involved then being

$$r \sim (I)R : r^{1+1t} s^{2m+2n-4} g(6m + 6n - 10) p f_{2,1}^k f_{2,k+1} \cdots f_{2,n-1},$$

$$s \sim (I)S : r^2 s^{2m+2n-5+(2m+n-4)t} g(4m - 8) l(6m + 6n - 10) p f_{2,1}^k f_{2,k+1} \cdots f_{2,n-1},$$

and the intersection

$$[RS] : r^2 s^{4m^2+8mn+4n^2-18m-18n+20} g(6m + 6n - 10) p f_{2,1}^k f_{2,k+1} \cdots f_{2,n-1}.$$

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